

# Subword Complexity and $k$ -Synchronization

Daniel Goč, Luke Schaeffer, and Jeffrey Shallit

School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1  
Canada  
{dgoč,l3schaeffer,shallit}@uwaterloo.ca

---

## Abstract

We show that the subword complexity function  $\rho_{\mathbf{x}}(n)$ , which counts the number of distinct factors of length  $n$  of a sequence  $\mathbf{x}$ , is  $k$ -synchronized in the sense of Carpi if  $\mathbf{x}$  is  $k$ -automatic. As an application, we generalize recent results of Goldstein. We give analogous results for the number of distinct factors of length  $n$  that are primitive words or powers. In contrast, we show that the function that counts the number of unbordered factors of length  $n$  is *not* necessarily  $k$ -synchronized for  $k$ -automatic sequences.

**Keywords and phrases** Automata, sequences,  $k$ -automatic, synchronized, subword complexity

## 1 Introduction

We are concerned with the representation of integers in base  $k$ , where  $k \geq 2$  is an integer. We let  $\Sigma_k = \{0, 1, 2, \dots, k-1\}$ , and we let  $(n)_k$  denote the canonical representation of  $n$  in base  $k$ , starting with the most significant digit, and *without* leading zeroes. If  $x \in \Sigma_k^*$ , we let  $[x]_k$  denote the integer represented by  $x$  (where  $x$  is allowed to have leading zeroes). To represent a pair of integers  $(m, n)$ , we use words over the alphabet  $\Sigma_k \times \Sigma_k$ . For such a word  $x$ , we let  $\pi_i(x)$  to be the projection onto the  $i$ 'th coordinate. The canonical representation  $(m, n)_k$  is defined to be the word  $x$  such that  $[\pi_1(x)]_k = m$  and  $[\pi_2(x)]_k = n$ , and having no leading  $[0, 0]$ 's. For example  $(43, 17)_2 = [1, 0][0, 1][1, 0][0, 0][1, 0][1, 1]$ .

Recently, Arturo Carpi and his co-authors [6, 4, 5] introduced a very interesting class of sequences that are computable by automata in a novel fashion: the class of  $k$ -synchronized sequences. Let  $(f(n))_{n \geq 0}$  be a sequence taking values in  $\mathbb{N}$ . They call such a sequence *k-synchronized* if there is a deterministic finite automaton  $M$  accepting the base- $k$  representation of the graph of  $f$ , namely  $\{(n, f(n))_k : n \geq 0\}$ .

Sequences that are  $k$ -synchronized are “halfway between” the class of  $k$ -automatic sequences, introduced by Cobham [9] and studied in many papers; and the class of  $k$ -regular sequences, introduced by Allouche and Shallit [1, 2]. They are particularly interesting for two reasons. If a sequence  $(f(n))$  is  $k$ -synchronized, then

- (a) we immediately get a bound on its growth rate:  $f(n) = O(n)$ ;
- (b) we immediately get a linear-time algorithm for efficiently calculating  $f(n)$ .

Result (a) can be found in [6, Prop. 2.5]. We now state and prove result (b).

► **Theorem 1.** *Suppose  $(f(n))_{n \geq 0}$  is  $k$ -synchronized. Then there is an algorithm that, given the base- $k$  representation of  $n$ , will compute the base- $k$  representation of  $f(n)$  in  $O(\log n)$  time.*

**Proof.** We know there is a DFA  $M = (Q, \Sigma_k \times \Sigma_k, \delta, q_0, F)$  accepting  $L = \{(n, f(n))_k : n \geq 0\}$ . From result (a) above we know that  $f(n) \leq Cn$ , for some constant  $C$ , so if  $(n, f(n))_k$  is accepted, then the first component is  $0^s w$  for some  $s \leq \log_k C$ , where  $w$  is the canonical base- $k$  representation of  $n$ . Let  $N = s + |w|$ . We now create a directed graph out of  $N + 1$  copies of the transition graph for  $M$ , by starting at the final states of  $M$  and tracing a path



backwards, using the reversed transitions of  $M$ . This path is chosen so the first component of the labels encountered form  $w^R 0^s$  and the second component is arbitrary. The resulting graph has at most  $O(N)$  transitions and vertices. There will be only one path of length  $l$  with  $|w| \leq l \leq N$  that leads to the initial state  $q_0$ , and this can be found with depth-first search in  $O(N)$  time. Then, reading the corresponding labels of the second components in the forward direction gives the base- $k$  representation of  $f(n)$ . ◀

In this paper, we are concerned with infinite words over a *finite* alphabet. Let  $\mathbf{x} = a_0 a_1 a_2 \dots$  be an infinite word. By  $\mathbf{x}[m..n]$  we mean the factor  $a_m a_{m+1} \dots a_n$  of  $\mathbf{x}$  of length  $n - m + 1$ . The subword complexity function  $\rho_{\mathbf{x}}(n)$  counts the number of distinct factors of length  $n$ .

An infinite word or sequence  $\mathbf{x}$  is said to be  $k$ -automatic if there is an automaton with outputs associated with the states that, on input  $(n)_k$ , reaches a state with output  $\mathbf{x}[n]$ . In this paper we show that if  $\mathbf{x}$  is a  $k$ -automatic sequence, then the subword complexity  $\rho_{\mathbf{x}}(n)$  is  $k$ -synchronized. As an application, we generalize and simplify recent results of Goldstein [13, 14]. Furthermore, we obtain analogous results for the number of length- $n$  primitive words and the number of length- $n$  powers.

We remark that there are a number of quantities about  $k$ -automatic sequences already known to be  $k$ -synchronized. These include

- the separator sequence of a non-ultimately-periodic  $k$ -automatic sequence [6];
- the repetitivity index of a  $k$ -automatic sequence [4];
- the recurrence function of a  $k$ -automatic sequence [8];
- the “appearance” function of a  $k$ -automatic sequence [8].

The latter two examples were not explicitly stated to be  $k$ -synchronized in [8], but the result follows immediately from the proofs in that paper.

## 2 Subword complexity

Cobham [9] proved that if  $\mathbf{x}$  is a  $k$ -automatic sequence, then  $\rho_{\mathbf{x}}(n) = O(n)$ . Cassaigne [7] proved that any infinite word  $\mathbf{x}$  satisfying  $\rho_{\mathbf{x}}(n) = O(n)$  also satisfies  $\rho_{\mathbf{x}}(n+1) - \rho_{\mathbf{x}}(n) = O(1)$ . Carpi and D’Alonzo [5] showed that the subword complexity function  $\rho_{\mathbf{x}}(n)$  is a  $k$ -regular sequence.

Charlier, Rampersad, and Shallit [8] found this result independently, using a somewhat different approach. They used the following idea. Call an occurrence of the factor  $t = \mathbf{x}[i..i+n-1]$  “novel” if  $t$  does not appear as a factor of  $\mathbf{x}[0..i+n-2]$ . In other words, the leftmost occurrence of  $t$  in  $\mathbf{x}$  is at position  $i$ . Then the number of factors of length  $n$  in  $\mathbf{x}$  is equal to the number of novel occurrences of factors of length  $n$ . The property that  $\mathbf{x}[i..i+n-1]$  is novel can be expressed as a predicate, as follows:

$$\begin{aligned} \{(n, i)_k : \forall j, 0 \leq j < i \ \mathbf{x}[i..i+n-1] \neq \mathbf{x}[j..j+n-1]\} = \\ \{(n, i)_k : \forall j, 0 \leq j < i \ \exists m, 0 \leq m < n \ \mathbf{x}[i+m] \neq \mathbf{x}[j+m]\}. \end{aligned} \quad (1)$$

As shown in [8], the base- $k$  representation of the integers satisfying any predicate of this form (expressible using quantifiers, integer addition and subtraction, indexing into a  $k$ -automatic sequence  $\mathbf{x}$ , logical operations, and comparisons) can be accepted by an explicitly-constructable deterministic finite automaton. From this, it follows that the sequence  $\rho_{\mathbf{x}}(n)$  is  $k$ -regular, and hence can be computed explicitly in terms of the product of certain matrices and vectors depending on the base- $k$  expansion of  $n$ .

We show that, in fact, the subword complexity function  $\rho_{\mathbf{x}}(n)$  is  $k$ -synchronized. The main observation needed is the following (Theorem 3): in any sequence of linear complexity, the novel occurrences of factors are “clumped together” in a bounded number of contiguous blocks. This makes it easy to count them.

More precisely, let  $\mathbf{x}$  be an infinite word and for any  $n$  consider the set of novel occurrences

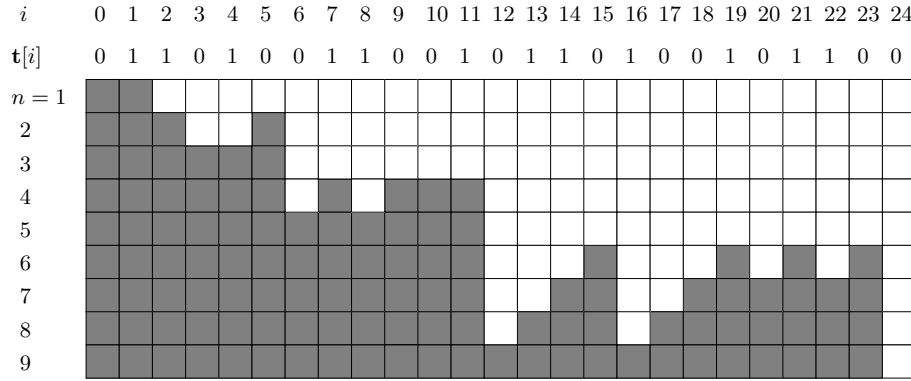
$$E_{\mathbf{x}}(n) := \{i : \text{the occurrence } \mathbf{x}[i..i+n-1] \text{ is novel}\}.$$

We consider how  $E_{\mathbf{x}}(n)$  evolves with increasing  $n$ .

As an example, consider the Thue-Morse sequence

$$\mathbf{t} = t_0 t_1 t_2 \cdots = 0110100110010110 \cdots,$$

defined by letting  $t_n$  be the number of 1's in the binary expansion of  $n$ , taken modulo 2. The gray squares in the rows of of Figure 1 depict the members of  $E_{\mathbf{t}}(n)$  for the Thue-Morse sequence for  $1 \leq n \leq 9$ .



■ **Figure 1** Evolution of novel occurrences of factors in the Thue-Morse sequence

► **Lemma 2.** *Let  $\mathbf{x}$  be an infinite word. If the factor of length  $n$  beginning at position  $i$  is a novel occurrence, so is*

- (a) *the factor of length  $n+1$  beginning at position  $i$ ;*
- (b) *the factor of length  $n+1$  beginning at position  $i-1$  (for  $i \geq 1$ ).*

**Proof.** (a) Suppose the factor of length  $n+1$  also occurs at some position  $j < i$ . Then the factor of length  $n$  also occurs at position  $j$ , contradicting the fact that it was a novel occurrence at  $i$ .

(b) Suppose the factor of length  $n+1$  beginning at position  $i-1$  occurs at some earlier position  $j < i-1$ . We can write the factor as  $ax$ , where  $a$  is a single letter and  $x$  is a word, so the factor of length  $n$  beginning at position  $i$  must also occur at position  $j+1 < i$ . But then it is not a novel occurrence. ◀

► **Theorem 3.** *Let  $\mathbf{x}$  be an infinite word. For  $n \geq 1$ , the number of contiguous blocks in  $E_{\mathbf{x}}(n)$  is at most  $\rho_{\mathbf{x}}(n) - \rho_{\mathbf{x}}(n-1) + 1$ .*

**Proof.** We prove the claim by induction on  $n$ . For  $n = 1$  the claim says there are at most  $\rho_{\mathbf{x}}(1)$  contiguous blocks, which is evidently true, since there are at most  $\rho_{\mathbf{x}}(1)$  novel factors of length 1.

Now assume the claim is true for all  $n' < n$ ; we prove it for  $n$ . Consider the evolution of the novel occurrences of factors in going from length  $n - 1$  to  $n$ . Every occurrence that was previously novel is still novel, and furthermore in every contiguous block except the first, we get novel occurrences at one position to the left of the beginning of the block. So if row  $n - 1$  has  $t$  contiguous blocks, then we get  $t - 1$  novel occurrences at the beginning of each block, except the first. (Of course, the first block begins at position 0, since any factor beginning at position 0 is novel, no matter what the length is.) The remaining  $\rho(n) - \rho(n - 1) - (t - 1)$  novel occurrences could be, in the worst case, in their own individual contiguous blocks. Thus row  $n$  has at most  $t + \rho(n) - \rho(n - 1) - (t - 1) = \rho(n) + \rho(n - 1) + 1$  contiguous blocks.  $\blacktriangleleft$

In our Thue-Morse example, it is well-known that  $\rho_t(n) - \rho_t(n - 1) \leq 4$ , so the number of contiguous blocks in any row is at most 5. This is achieved, for example, for  $n = 6$ .

► **Example 4.** We give an example of a recurrent infinite word over a finite alphabet where the number of contiguous blocks in  $E_{\mathbf{x}}(n)$  is unbounded. Consider the word

$$\mathbf{w} = \prod_{n \geq 1} (n)_2 = 110111001011101111000 \dots$$

Then for each  $n \geq 5$  the first occurrence of each of the words  $0^{n-1}1, 0^{n-2}11, \dots, 0^21^{n-2}$  have a non-novel occurrence immediately following them, which shows there are at least  $n - 2$  blocks in  $E_{\mathbf{w}}(n)$ .

► **Corollary 5.** *If  $\rho_{\mathbf{x}}(n) = O(n)$ , then there is a constant  $C$  such that every row  $E_{\mathbf{x}}(n)$  in the evolution of novel occurrences consists of at most  $C$  contiguous blocks.*

**Proof.** By the result of Cassaigne [7], we know that there exists a constant  $C$  such that  $\rho_{\mathbf{x}}(n) - \rho_{\mathbf{x}}(n - 1) \leq C - 1$ . By Theorem 3, we know there are at most  $C$  contiguous blocks in any  $E_{\mathbf{x}}(n)$ .  $\blacktriangleleft$

► **Theorem 6.** *Let  $\mathbf{x}$  be a  $k$ -automatic sequence. Then its subword complexity function  $\rho_{\mathbf{x}}(n)$  is  $k$ -synchronized.*

**Proof.** Following [8], it suffices to show how to accept the language

$$\{(n, m)_k : n \geq 0 \text{ and } m = \rho_{\mathbf{x}}(n)\}$$

with a finite automaton. Here is a sketch of the argument. From our results above, we know that there is a finite constant  $C \geq 1$  such that the number of contiguous blocks in any row of the factor evolution diagram is bounded by  $C$ . So we simply “guess” the endpoints of every block and then verify that each factor of length  $n$  starting at the positions inside blocks is a novel occurrence, while all other factors are not. Finally, we verify that  $m$  is the sum of the sizes of the blocks.

To fill in the details, we observe above in (1) that the predicate “the factor of length  $n$  beginning at position  $i$  of  $\mathbf{x}$  is a novel occurrence” is solvable by a finite automaton. Similarly, given endpoints  $a, b$  and  $n$ , the predicates “every factor of length  $n$  beginning at positions  $a$  through  $b$  is a novel occurrence” and “no factor of length  $n$  beginning at positions  $a$  through  $b$  is a novel occurrence” are also solvable by a finite automaton. The length of each block is just  $b - a + 1$ , and it is easy to create an automaton that will check if the sums of the lengths of the blocks equals  $m$ , which is supposed to be  $\rho_{\mathbf{x}}(n)$ .  $\blacktriangleleft$

Applying Theorem 1 we get

► **Corollary 7.** *Given a  $k$ -automatic sequence  $\mathbf{x}$ , there is an algorithm that, on input  $n$  in base  $k$ , will produce  $\rho_{\mathbf{x}}(n)$  in base  $k$  in time  $O(\log n)$ .*

As another application, we can recover and improve some recent results of Goldstein [13, 14]. He showed how to compute the quantities  $\limsup_{n \geq 1} \rho_{\mathbf{x}}(n)/n$  and  $\liminf_{n \geq 1} \rho_{\mathbf{x}}(n)/n$  for the special case of  $k$ -automatic sequences that are the fixed points of  $k$ -uniform morphisms related to certain groups. Corollary 8 below generalizes these results to all  $k$ -automatic sequences.

► **Corollary 8.** *There is an algorithm, that, given a  $k$ -automatic sequence  $\mathbf{x}$ , will compute  $\sup_{n \geq 1} \rho_{\mathbf{x}}(n)/n$ ,  $\limsup_{n \geq 1} \rho_{\mathbf{x}}(n)/n$ , and  $\inf_{n \geq 1} \rho_{\mathbf{x}}(n)/n$ ,  $\liminf_{n \geq 1} \rho_{\mathbf{x}}(n)/n$ .*

**Proof.** We already showed how to construct an automaton accepting  $\{(n, \rho_{\mathbf{x}}(n))_k : n \geq 1\}$ . Now we just use the results from [17, 16]. Notice that the  $\limsup$  corresponds to what is called the largest “special point” in [16]. ◀

► **Example 9.** Continuing our example of the Thue-Morse sequence, Figure 2 displays a DFA accepting

$$\{(n, \rho_{\mathbf{t}}(n))_k : n \geq 0\}.$$

Inputs are given with the most significant digit first; the “dead” state and transitions leading to it are omitted.

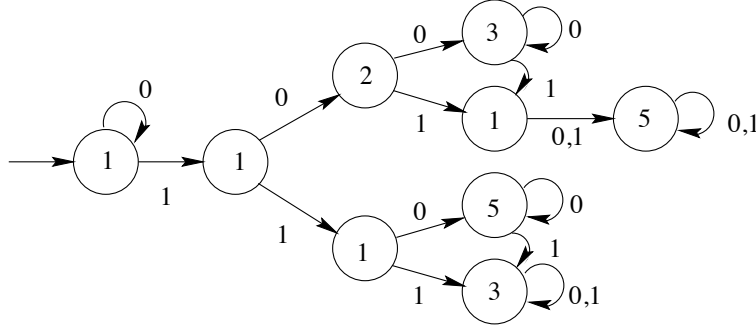
Given an infinite word  $\mathbf{x}$ , we can also count the number of contiguous blocks in each  $E_{\mathbf{x}}(n)$  for  $n \geq 0$ . (For the Thue-Morse sequence this gives the sequence 1, 1, 2, 1, 3, 1, 5, 3, 3, 1, ...) If  $\mathbf{x}$  is  $k$ -automatic, then this sequence is also, as the following theorem shows:

► **Theorem 10.** *If  $\mathbf{x}$  is  $k$ -automatic then the sequence  $(e(n))_{n \geq 0}$  counting the number of contiguous blocks in the  $n$ 'th step  $E_{\mathbf{x}}(n)$  of the evolution of novel occurrences of factors in  $\mathbf{x}$  is also  $k$ -automatic.*

**Proof.** Since we have already shown that the number of contiguous blocks is bounded by some constant  $C$  if  $\mathbf{x}$  is  $k$ -automatic, it suffices to show for each  $i \leq C$  we can create an automaton to accept the language  $\{(n)_k : E_{\mathbf{x}}(n) \text{ has exactly } i \text{ contiguous blocks}\}$ . To do so, on input  $n$  in base  $k$  we guess the endpoints of the  $i$  contiguous nonempty blocks, verify that the length- $n$  occurrences at those positions are novel, and that all other occurrences are not novel. ◀



4. If  $M_{j+1} = M_j$  then we are done. We create an automaton that accepts  $(n, S)$  if there exists  $e$  such that  $M_j$  accepts  $(n, S, e)$ .



■ **Figure 3** Automaton computing number of contiguous blocks of novel occurrences of length- $n$  factors in the Thue-Morse sequence

Besides the automaton depicted in Figure 1, we ran our program on the paperfolding sequence [11] and the so-called “period-doubling sequence” [10]. The results are depicted below in Figures 4 and 5.

#### 4 Powers and primitive words

Let  $w$  be a nonempty word. We say  $w$  is a *power* if there exists a word  $x$  and an integer  $k \geq 2$  such that  $w = x^k$ ; otherwise we say  $w$  is *primitive*. Given a word  $z$ , there is a unique way to write it as  $y^i$ , where  $y$  is primitive and  $i$  is an integer  $\geq 1$ ; this  $y$  is called the *primitive root* of  $z$ . Thus, for example, the primitive root of **murmur** is **mur**.

We say  $w = a_1 \cdots a_n$  has a *period*  $p$  if  $a_i = a_{i+p}$  for  $1 \leq i \leq n - p$ . Thus, for example, **alfalfa** has period 3. It is easy to see that a word  $w$  is a power if and only if it has a period  $p$  such that  $p < |w|$  and  $p \mid |w|$ .

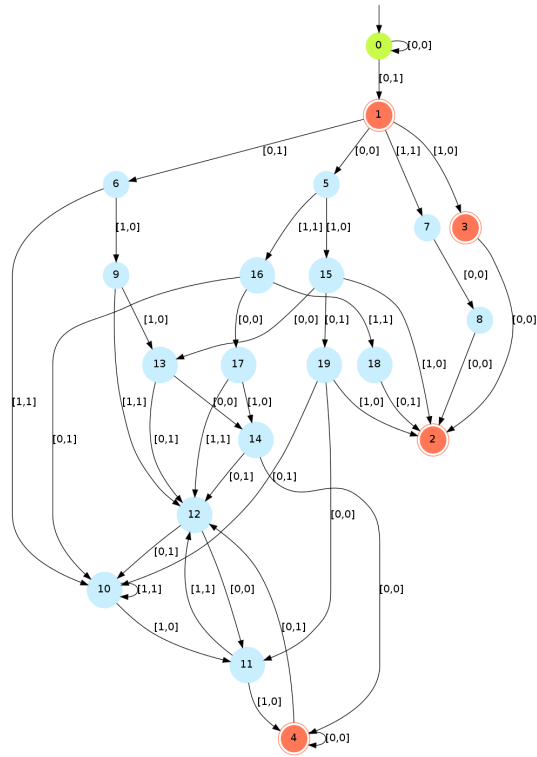
Two finite words  $x, y$  are conjugates if one is a cyclic shift of the other; in other words, if there exist words  $u, v$  such that  $x = uv$  and  $y = vu$ . For example, **enlist** is a conjugate of **listen**. As is well-known, every conjugate of a power of a word  $x$  is a power of a conjugate of  $x$ . The lexicographically least conjugate of a primitive word is called a *Lyndon word*. We call the lexicographically least conjugate of the primitive root of  $x$  the *Lyndon root* of  $x$ .

The following lemma says that if we consider the starting positions of length- $n$  powers in a word  $x$ , then there must be large gaps between contiguous blocks of such starting positions.

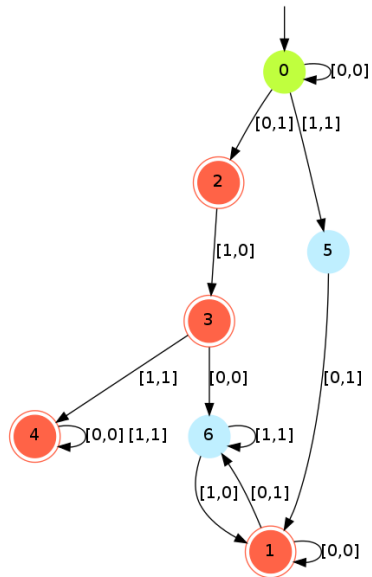
► **Lemma 12.** *Let  $z$  be a finite or infinite word, and let  $n \geq 2$  be an integer. Suppose there exist integers  $i, j$  such that*

- (a)  $w_1 := z[i..i + n - 1]$  is a power;
- (b)  $w_2 := z[j..j + n - 1]$  is a power;
- (c)  $i < j \leq i + n/3$ .

*Then  $z[t..t + n - 1]$  is a power for  $i \leq t \leq j$ . Furthermore, if  $x_1$  is the Lyndon root of  $w_1$ , then  $x_1$  is also the Lyndon root of each word  $z[t..t + n - 1]$ .*



■ **Figure 4** Automaton computing the subword complexity of the paperfolding sequence



■ **Figure 5** Automaton computing the subword complexity of the period-doubling sequence



**Proof.** Let  $x_1$  be the primitive root of  $w_1$  and  $x_2$  be the primitive root of  $w_2$ . Since  $x_1$  and  $x_2$  are powers, there exist integers  $p_1, p_2 \geq 2$  such that  $w_1 = x_1^{p_1}$  and  $w_2 = x_2^{p_2}$ .

Since  $w_1$  and  $w_2$  are both of length  $n$ , and since their starting positions are related by  $i < j \leq i + n/3$ , it follows that the word  $v := z[j..i + n - 1]$  is common to both  $w_1$  and  $w_2$ , and  $|v| = i + n - j \geq i + 2n/3 + n/3 - j \geq 2n/3$ .

Now there are three cases to consider:

- (a)  $|x_1| > |x_2|$ ;
- (b)  $|x_1| < |x_2|$ ;
- (c)  $|x_1| = |x_2|$ .

Case (a): We must have  $p_2 > p_1 \geq 2$ , so  $p_2 \geq 3$ . Since  $v$  is a suffix of  $w_1$ , it has period  $|x_1| \leq n/2$ . Since  $v$  is a prefix of  $w_2$ , it has period  $|x_2| \leq n/3$ . Then  $|v| \geq 2n/3 \geq |x_1| + |x_2|$ . By a theorem of Fine and Wilf [12], it now follows that  $v$ , and hence  $x_1$ , has period  $p := \gcd(|x_1|, |x_2|) \leq |x_2| < |x_1|$ . Now  $p$  is less than  $|x_1|$  and also divides it, so this means  $x_1$  is a power, a contradiction, since we assumed  $x_1$  is primitive. So this case cannot occur.

Case (b) gives a similar contradiction.

Case (c): We have  $p_1 = p_2 \geq 2$ . Then the last occurrence of  $x_1$  in  $w_1$  lies inside  $x_2^2$ , and so  $x_1$  is a conjugate of  $x_2$ . Hence  $w_1$  is a conjugate of  $w_2$ . It now follows that  $z[t..t + n - 1]$  is a conjugate of  $w_1$  for every  $t$ ,  $i \leq t \leq j$ . But the conjugate of a power is itself a power, and we are done.  $\blacktriangleleft$

► **Remark.** The bound of  $n/3$  in the statement of Lemma 12 is best possible, as shown by the following class of examples. Let  $h$  be the morphism that maps  $1 \rightarrow 21$  and  $2 \rightarrow 22$ , and consider the word

$$h^i(122122121212).$$

This word is of length  $12 \cdot 2^i$ , and contains squares of length  $3 \cdot 2^{i+1}$  starting in the first  $3 \cdot 2^i$  positions, and cubes of length  $3 \cdot 2^{i+1}$  ending in the last  $2^i + 1$  positions. This achieves a gap of  $n/3 + 1$  infinitely often.

Now, given an infinite word  $\mathbf{x}$ , we define a function  $\alpha_{\mathbf{x}}(n)$ , the *appearance function*, to be the least index  $i$  such that every length- $n$  factor of  $\mathbf{x}$  appears in the prefix  $\mathbf{x}[0..i + n - 1]$ ; see [3, §10.10].

► **Theorem 13.** *If  $\mathbf{x}$  is a  $k$ -automatic sequence, then  $\alpha_{\mathbf{x}}(n) = O(n)$ .*

**Proof.** First, we show that the appearance function is  $k$ -synchronized. It suffices to show that there is an automaton accepting  $\{(n, m)_k : m = \alpha_{\mathbf{x}}(n)\}$ . To see this, note that on input  $(n, m)_k$  we can check that

- for all  $i \geq 0$  there exists  $j$ ,  $0 \leq j \leq m$  such that  $\mathbf{x}[i..i + n - 1] = \mathbf{x}[j..j + n - 1]$ ; and
- for all  $l < m$  we have  $\mathbf{x}[m..m + n - 1] \neq \mathbf{x}[l..l + n - 1]$ .

From [6, Prop. 2.5] we know  $k$ -synchronized functions are  $O(n)$ .  $\blacktriangleleft$

As before, we consider maximal blocks of novel occurrences of length- $n$  powers in  $\mathbf{x}$ . Our goal is to prove

► **Lemma 14.** *If  $\mathbf{x}$  is  $k$ -automatic, then there are only a constant number of such blocks.*

**Proof.** To begin with, we consider maximal blocks of length- $n$  powers in  $\mathbf{x}$  (not considering whether they are novel occurrences). From Theorem 13 we know that every length- $n$  factor must occur at a position  $< Cn$ , for some constant  $C$  (depending on  $\mathbf{x}$ ). We first argue that

the number of maximal blocks of length- $n$  powers, up to the position of the last length- $n$  power to occur for the first time, is at most  $3C$ .

Suppose there  $\geq 3C + 1$  such blocks. Then Lemma 12 says that any two such blocks must be separated by at least  $n/3$  positions. So the first occurrence of the last factor to occur occurs at a position  $\geq (3C)(n/3) = Cn$ , a contradiction.

So using a constant number of blocks, in which each position of each block starts a length- $n$  factor that is a power, we cover the starting positions of all such factors. It now remains to process these blocks to remove occurrences of length- $n$  powers that are not novel.

The first thing we do is remove from each block the positions starting length- $n$  factors that have already occurred *in that block*. This has the effect of truncating long blocks. The new blocks have the property that each factor occurring at the starting positions in the blocks never appeared before in that block.

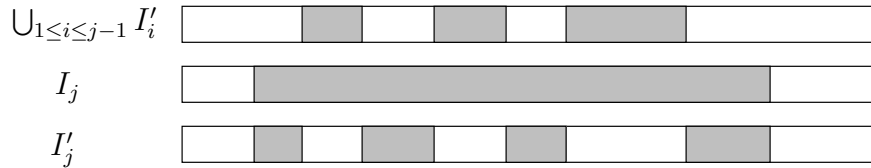
Above we already proved that inside each block, the powers that begin at each position are all powers of some conjugate of a fixed Lyndon word. Now we process the blocks associated with the same Lyndon root together, from the first (leftmost) to the last. At each step, we remove from the current block all the positions where length- $n$  factors begin that have appeared in any previous block. When all blocks have been processed, we need to see that there are still at most a constant number of contiguous blocks remaining.

Suppose the associated Lyndon root is  $y$ , with  $|y| = d$ . Each position in a block is the starting position of a power of a conjugate of  $y$ , and hence corresponds to a right rotation of  $y$  by some integer  $i$ ,  $0 \leq i < d$ . Thus each block  $B_j$  actually corresponds to some  $I_j$  that is a contiguous subblock of  $0, 1, \dots, d-1$  (thought of as arranged in a circle).

As we process the blocks associated with  $y$  from left to right we replace  $I_j$  with  $I'_j := I_j - (I_1 \cup \dots \cup I_{j-1})$ . Now if  $I \subseteq \{0, 1, \dots, d-1\}$  is a union of contiguous subblocks, let  $\#I$  be the number of contiguous subblocks making up  $I$ . We claim that

$$\#I'_1 + \#I'_2 + \dots + \#I'_n + \#(\bigcup_{1 \leq i \leq n} I'_i) \leq 2n. \quad (2)$$

To see this, suppose that when we set  $I'_n := I_n - (I_1 \cup \dots \cup I_{n-1})$ , the subblock  $I_n$  has an intersection with  $t$  of the lower-numbered subblocks. Forming the union  $(\bigcup_{1 \leq i \leq n} I'_i)$  then obliterates  $t$  subblocks and replaces them with 1. But  $I'_n$  has  $t-1$  new subblocks, plus at most 2 at either edge (see Figure 6). This means that the left side of (2) increases by at most  $(1-t) + (t-1) + 2 = 2$ . Doing this  $n$  times gives the result.



■ **Figure 6** How the number of blocks changes

Now at the end of the procedure there will be at least one interval in the union of all the  $I_i$ , so  $\#I'_1 + \#I'_2 + \dots + \#I'_n \leq 2n - 1$ , and we have proved (2).

Earlier we showed that there are at most  $3C$  maximal blocks of length- $n$  powers, up to the position of the last length- $n$  power to occur for the first time. Then, after processing these blocks to remove positions corresponding to factors that occurred earlier, we will have at most  $2(3C) = 6C$  blocks remaining. ◀

► **Corollary 15.** *If  $\mathbf{x}$  is  $k$ -automatic, then*

- *the function counting the number of distinct length- $n$  factors that are powers is  $k$ -synchronized;*
- *the function counting the number of distinct length- $n$  factors that are primitive words is  $k$ -synchronized.*

**Proof.** Suppose  $\mathbf{x}$  is  $k$ -automatic, and generated by the DFAO  $M$ . From the Lyndon-Schützenberger theorem [15], we know that a word  $x$  is a power if and only if there exist nonempty words  $y, z$  such that  $x = yz = zy$ . Thus, we can express the predicate  $P(i, j) := “\mathbf{x}[i..j] \text{ is a power}”$  as follows: “there does not exist  $d$ ,  $0 < d < j - i + 1$ , such that  $\mathbf{x}[i..j - d] = \mathbf{x}[i + d..j]$  and  $\mathbf{x}[j - d + 1..j] = \mathbf{x}[i..i + d - 1]”$ . Furthermore, we can express the predicate  $P'(i, n) := “\mathbf{x}[i..i + n - 1] \text{ is a length-}n \text{ power and the first occurrence of that power in } \mathbf{x}”$ , as

$$P(i, i + n - 1) \wedge (\forall i', 0 \leq i' < i, \neg P(i', i' + n - 1)).$$

From Lemma 14 we know that the novel occurrences of length- $n$  powers are clustered into a finite number of blocks. Then, as in the proof of Theorem 6, we can guess the endpoints of these blocks, and verify that the length- $n$  factors beginning at the positions inside the blocks are novel occurrences of powers, while those outside are not, and sum the lengths of the blocks, using a finite automaton built from  $M$ . Thus, the function counting the number of length- $n$  powers in  $\mathbf{x}$  is  $k$ -synchronized.

The number of length- $n$  primitive words in  $\mathbf{x}$  is then also  $k$ -synchronized, since it is expressible as the total number of words of length  $n$ , minus the number of length- $n$  powers. ◀

► **Remark.** Using the technique above, we can prove analogous results for the functions counting the number of length- $n$  words that are  $\alpha$ -powers, for any fixed rational number  $\alpha > 1$ .

## 5 Unsynchronized sequences

It is natural to wonder whether other aspects of  $k$ -automatic sequences are always  $k$ -synchronized. We give an example that is not.

We say a word  $w$  is *bordered* if it has a nonempty prefix, other than  $w$  itself, that is also a suffix. Alternatively,  $w$  is bordered if it can be written in the form  $w = tv$ , where  $t$  is nonempty. Otherwise a word is *unbordered*.

Charlier et al. [8] showed that  $u_{\mathbf{x}}(n)$ , the number of unbordered factors of length  $n$  of a sequence  $\mathbf{x}$ , is  $k$ -regular if  $\mathbf{x}$  is  $k$ -automatic. They also gave a conjecture for recursion relations defining  $u_{\mathbf{t}}(n)$  where  $\mathbf{t}$  is the Thue-Morse sequence; this conjecture has recently been verified by Goč and Shallit.

We give here an example of a  $k$ -automatic sequence where the number of unbordered factors of length  $n$  is not  $k$ -synchronized.

Consider the characteristic sequence of the powers of 2:

$$\mathbf{c} := 011010001000000010 \dots$$

► **Theorem 16.** *The sequence  $\mathbf{c}$  is 2-automatic, but the function  $u_{\mathbf{c}}(n)$  counting the number of unbordered factors is not 2-synchronized.*

**Proof.** It is not hard to verify that  $\mathbf{c}$  is 2-automatic and that  $\mathbf{c}$  has exactly  $r + 2$  unbordered factors of length  $2^r + 1$ , for  $r \geq 2$  — namely, the factors beginning at positions  $2^i$  for  $0 \leq i \leq r$ , and the factor beginning at position  $2^r + 1$ . However, if  $u_{\mathbf{c}}(n)$  were 2-synchronized, then reading an input where the first component looks like  $0^i 10^r 1$  (and hence a representation of  $2^{r-1} + 1$ ) for large  $r$  would force the transitions to enter a cycle. If the corresponding transitions for the second component contained a nonzero entry, this would force  $u_{\mathbf{c}}(n)$  to grow linearly with  $n$  when  $n$  is of the form  $2^r + 1$ . Otherwise, the corresponding transitions for the second component are just 0's, in which case  $u_{\mathbf{c}}(n)$  is bounded above by a constant, for  $n$  of the form  $2^r + 1$ . Both cases lead to a contradiction. ◀

---

### References

- 1 J.-P. Allouche and J. Shallit. The ring of  $k$ -regular sequences. *Theoret. Comput. Sci.* **98** (1992), 163–197.
- 2 J.-P. Allouche and J. Shallit. The ring of  $k$ -regular sequences, II. *Theoret. Comput. Sci.* **307** (2003), 3–29.
- 3 J.-P. Allouche and J. Shallit. *Automatic Sequences: Theory, Applications, Generalization*. Cambridge University Press, 2003.
- 4 A. Carpi and V. D'Alonzo. On the repetitivity index of infinite words. *Internat. J. Algebra Comput.* **19** (2009), 145–158.
- 5 A. Carpi and V. D'Alonzo. On factors of synchronized sequences. *Theoret. Comput. Sci.* **411** (2010), 3932–3937.
- 6 A. Carpi and C. Maggi. On synchronized sequences and their separators. *RAIRO Inform. Théor. App.* **35** (2001), 513–524.
- 7 J. Cassaigne. Special factors of sequences with linear subword complexity. In J. Dassow, G. Rozenberg, and A. Salomaa, editors, *Developments in Language Theory II*, pp. 25–34. World Scientific, 1996.
- 8 E. Charlier, N. Rampersad, and J. Shallit. Enumeration and decidable properties of automatic sequences. In G. Mauri and A. Leporati, editors, *Developments in Language Theory—15th International Conference, DLT 2011*, Vol. 6795, pp. 165–179. Springer-Verlag, 2011.
- 9 A. Cobham. Uniform tag sequences. *Math. Systems Theory* **6** (1972), 164–192.
- 10 D. Damanik. Local symmetries in the period-doubling sequence. *Disc. Appl. Math.* **100** (2000), 115–121.
- 11 F. M. Dekking, M. Mendès France, and A. J. van der Poorten. Folds! *Math. Intelligencer* **4** (1982), 130–138, 173–181, 190–195. Erratum, **5** (1983), 5.
- 12 N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.
- 13 I. Goldstein. Asymptotic subword complexity of fixed points of group substitutions. *Theoret. Comput. Sci.* **410** (2009), 2084–2098.
- 14 I. Goldstein. Subword complexity of uniform D0L words over finite groups. *Theoret. Comput. Sci.* **412** (2011), 5728–5743.
- 15 R. C. Lyndon and M. P. Schützenberger. The equation  $a^M = b^N c^P$  in a free group. *Michigan Math. J.* **9** (1962), 289–298.
- 16 L. Schaeffer and J. Shallit. The critical exponent is computable for automatic sequences. *Int. J. Found. Comput. Sci.*, to appear, 2012.
- 17 J. Shallit. The critical exponent is computable for automatic sequences. In P. Ambroz, S. Holub, and Z. Másaková, editors, *Proceedings 8th International Conference Words 2011*, Vol. 63 of *Elect. Proc. Theor. Comput. Sci.*, pp. 231–239. 2011.